

Towards a Mori Theory on Compact Kähler Threefolds III

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Introduction

In this note we continue the study of the bimeromorphic geometry of compact Kähler threefolds. The final aim should be - as in the algebraic case - to construct minimal models for threefolds with non-negative Kodaira dimension, describe the way how to get a minimal model, to prove abundance for the minimal models, i.e. semi-ampleness of the canonical bundle, and finally to construction Fano fibrations on appropriate models of threefolds with $\kappa = -\infty$.

Concerning abundance we show that the canonical divisor of a minimal Kähler threefold X is good, i.e. $\kappa(X) = \nu(X)$, where $\nu(X)$ denotes the numerical Kodaira dimension, i.e. the largest number m such that $K_X^m \neq 0$; unless X is simple and non-Kummer. Here X is *simple*, if there is no positive-dimensional subvariety through the general point of X and X is *Kummer* if it has a bimeromorphic model which is the quotient of a torus by a finite group. These simple non-Kummer varieties are expected not to exist but this can be only a consequence of a completely developed minimal model theory in the Kähler case. By a result of Nakayama [Na87] (Kawamata in the algebraic case), it follows from $\kappa(X) = \nu(X)$ that K_X is semi-ample, i.e. some multiple mK_X is generated by global sections. So abundance holds on Kähler threefolds with the possible exception of simple non-Kummer threefolds.

Furthermore we prove, using essentially Part 1 [CP96] and Part 2 [Pe96] to this paper that a smooth compact Kähler threefold X with K_X not nef carries a contraction unless possibly X is simple and non-Kummer. The main steps in the proof are the following

- (1) Construction of some curve $C \subset X$ with $K_X \cdot C < 0$. We distinguish the case $\kappa(X) \geq 0$ and $\kappa(X) = -\infty$. In the first case we examine carefully a member in the linear system $|mK_X|$ to construct C , in the second we use a result of a recent joint paper of Campana and the author saying that X is uniruled unless X is simple. In that second case it is immediately clear that we can choose C rational.
- (2) Next we make C rational (this step works for all compact Kähler threefolds). Here we construct from C a non-splitting family of irreducible curve and examine its structure. The reason why this family exists is the deformation lemma of Ein-Kollár: a curve C in a smooth threefold with $K_X \cdot C < 0$ moves.
- (3) The last step is the construction of a contraction from a rational curve C with $K_X \cdot C < 0$. This was in large parts already done in [CP96] and [Pe96]; here we finish the study in sect. 1 of this paper.

We summarise the results of this paper in the following two theorems.

Theorem 1 *Let X be a minimal Kähler threefold (\mathbf{Q} -factorial with at most terminal singularities). Assume that X is not both simple and non-Kummer. Then $\kappa(X) = \nu(X)$, hence K_X is semi-ample. In particular, if $\kappa(X) = 0$ (and X not simple non-Kummer), then $K_X \equiv 0$ and $mK_X = \mathcal{O}_X$ for some positive m .*

Theorem 2 *Let X be a smooth compact Kähler threefold with K_X not nef. Then X carries a contraction unless (possibly) X is simple with $\kappa(X) = -\infty$.*

By a contraction we mean a surjective map $\varphi : X \rightarrow Y$ to a normal compact variety with connected fibers such that $-K_X$ is φ -ample and $b_2(X) = b_2(Y) + 1$. We would like to have that Y is again a Kähler space but at the moment we are still in trouble if φ is the blow-up of a smooth curve in the smooth threefold Y - in that case Y will be only Kähler if φ is chosen appropriately, namely the ray generated by the curves contracted by φ must be extremal in the dual cone to the Kähler cone of X (see [Pe96]).

There are several problems arising.

(a) First of all we would like to construct a curve with $K_X \cdot C < 0$ also in the "simple" case if K_X is not nef. This would mean that we should construct "directly" - i.e. without using any specific information on X some curve C with $K_X \cdot C < 0$. This requires certainly new techniques and probably a threefold proof would work in any dimension and also with terminal singularities.

(b) We need to prove the existence of contractions also for \mathbf{Q} -Gorenstein threefolds with terminal singularities in order to perform the Mori program. The Gorenstein case will probably be the same as the smooth case, but in the presence of non-Gorenstein singularities there are obstructions to moving curves so that some new arguments are needed.

(c) We must overcome the difficulty with the Kähler property of Y in case of a blow-up $\varphi : X \rightarrow Y$ along a smooth curve.

As already observed in [Pe96], one major consequence of this programme would be that simple Kähler threefolds are Kummer, in particular there are no simple threefolds of negative Kodaira dimension, all those being uniruled.

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Preliminaries

(0.1) Let X be an irreducible normal compact complex in class \mathcal{C} , e.g. a normal compact Kähler space. Let L be a line bundle on X . Then L is nef if there is a desingularisation $\pi : \hat{X} \rightarrow X$ such that $\pi^*(L)$ is nef. By [Pe96, 4.6] this is independent on the choice of π at least in dimension 3.

(0.2) A normal compact Kähler threefold (n -fold) is minimal if X is \mathbf{Q} -factorial, has only terminal singularities and K_X (i.e. some mK_X) is nef.

(0.3) We will often use $C_{n,m}$ for Kähler threefolds: if $f : X \rightarrow Y$ is a surjective fiber space with X a smooth compact Kähler threefold (so $n = 3$) then $\kappa(X) \geq \kappa(Y) + \kappa(F)$, where F denotes the general fiber ($m = \dim Y$). See [Fu78],[Ka81],[Ue87].

(0.4) A compact manifold is *simple* if there is no positive dimensional subvarieties through the general point of X . The only known Kähler examples arise from tori. To make this more precise one says that a compact manifold (or normal compact space) is *Kummer* if X is bimeromorphic to a quotient T/G of a torus by a finite group G . The conjecture is that simple Kähler manifolds are Kummer. A standard reference here is [Fu83].

(0.5) Some further notations: $a(X)$ denotes always the algebraic dimension of X . The irregularity of X is $q(X) = \dim H^1(X, \mathcal{O}_X)$. Finally $N_1(X) \subset H_2(X, \mathbf{R})$ is the vector space generated by the classes of irreducible curve. Inside $N_1(X)$ we have the closed cone $\overline{NE}(X)$ generated by the classes of irreducible curves.

1. Abundance for Kähler threefolds

In this section we prove the Abundance Conjecture for (non-simple) Kähler threefolds. First we show

1.1 Theorem *Let X be a minimal Kähler threefold with $\kappa(X) = 0$. Assume that X is not both simple and non-Kummer. Then $K_X \equiv 0$.*

Proof. The assertion being known for projective minimal threefolds by Miyaoka [Mi88] and Kawamata [Ka92], we shall assume that X is non-projective. Then by [CP98] X has a bimeromorphic model X' with at most quotient singularities such that there is a finite cover $\tilde{X} \rightarrow X'$, étale in codimension 1, with \tilde{X} a torus or a product $E \times S$ of an elliptic curve E with a K3-surface S . In the algebraic case such a conclusion is of course false: Calabi-Yau threefolds are simply connected. The reason why the non-algebraic case is somehow more special than the projective is the existence of 2-forms on non-algebraic threefolds (Kodaira's theorem). Now back in our specific situation, we conclude that $K_{X'} \equiv 0$. We claim that X' has only canonical singularities (a priori we know only that $(X, 0)$ has log terminal singularities). This is seen as follows. Choose $m > 0$ such that $mK_{X'} \simeq \mathcal{O}_{X'}$. Let $\pi : \hat{X} \rightarrow X$ be a desingularisation. Write

$$mK_{\hat{X}} = \pi^*(mK_{X'}) + \sum a_i E_i$$

with E_i the exceptional components of π . We need to show that $a_i \geq 0$ for all i . If some $a_i < 0$, then $\pi_*(mK_{\hat{X}})$ is a proper subsheaf of $mK_X = \mathcal{O}_X$; hence $H^0(mK_{\hat{X}}) = 0$. Passing to a high multiple of m , we deduce $\kappa(\hat{X}) = -\infty$, a contradiction. So X' has only canonical singularities. Take a partial crepant resolution $\pi : \hat{X} \rightarrow X'$ [Re80]. One can even assure that \hat{X} is \mathbf{Q} -factorial [Re83],[Ka88]. So \hat{X} has only terminal singularities and still $K_{\hat{X}} \equiv 0$. Now the bimeromorphic meromorphic map $X \dashrightarrow \hat{X}$ is an isomorphism in codimension 1 [Ha87,Ko89]. Here we use the fact that K_X is nef! Thus $K_X \equiv 0$.

1.2 Remark. In case $q(X) > 0$, it is much easier to conclude; the argument being independent of [CP98]. Here is the reasoning in that case. Let $\alpha : X \rightarrow A$ be the Albanese map of X . Since $\kappa(X) = 0$, α has to be surjective (by $C_{3,1}$ and $C_{3,2}$). Let $f : X \rightarrow Y$ be the Stein factorisation of α (we shall see that actually α has connected fibers). Notice that $\kappa(F) = 0$ for the general fiber F of f .

(1) Suppose $\dim A = \dim Y = 3$. Then $X \rightarrow A$ is unramified in codimension 1, in fact, otherwise K_X would contain the ramification divisor R whose image in A is a nef divisor with $\kappa > 0$, so that $\kappa(R) > 0$, hence $\kappa(X) > 0$. Of course, this is well-known, at least in the algebraic case, see [KV80]. So $\alpha = f$ (by the universal property of α), hence α is birational. Therefore

$$K_X = \alpha^*(K_A) + \sum_{i \in I} \lambda_i E_i = \sum \lambda_i E_i,$$

where E_i are the exceptional components of α and $\lambda_i > 0$. Then K_X being nef, it follows easily that $I = \emptyset$. So $K_X = \mathcal{O}_X$.

(2) Next suppose that $\dim Y = 1$. Hence $Y = A$ is an elliptic curve and again $f = \alpha$. Since K_F is nef and $\kappa(F) = 0$, we deduce that K_F is torsion. Choose $m > 0$ such that mK_X is Cartier and that $h^0(mK_X) = 1$. In particular $mK_F = \mathcal{O}_F$. Let $s \in H^0(mK_X)$ be a non-zero section. Then $s|_F = 0$ or $s|_F$ has no zeroes. Writing

$$mK_X = \sum a_i D_i$$

with $a_i > 0$, we conclude that $D_i \cap F = \emptyset$, i.e. $\dim D_i = 0$. On the other hand $\sum a_i D_i$ is α -nef, and this is only possible if some multiple kmK_X is a multiple of fibers so that

$$kmK_X = \alpha^*(L)$$

with a line bundle L on A . Since $\kappa(X) = 0$, we have $\kappa(L) = 0$, therefore $L = \mathcal{O}_A$.

(3) Finally, let $\dim Y = 2$. Again we conclude $f = \alpha$ and $Y = A$.

(3.1) $a(A) = 0$. Letting s and $mK_X = \sum a_i E_i$ be as in (2), we again have $\dim \alpha(D_i) \leq 1$ (now F is an elliptic curve). Since A does not contain compact curves, we have $\dim \alpha(D_i) = 0$ for all i , which again contradicts the nefness of K_X .

(3.2) $a(A) = 1$. Then we use the algebraic reduction $A \rightarrow B$ to an elliptic curve and obtain a map $g : X \rightarrow B$. Now we argue as in (2).

(3.3) $a(A) = 2$. By $C_{3,2}^+$ (cp. [Ue87]) we have $\text{Var}(f) = 0$. Hence there exists a finite cover $\tilde{A} \rightarrow A$ and a meromorphic generically finite map $F \times \tilde{A} \rightarrow X$ with an elliptic curve F . Thus X is algebraic, contradiction. So this subcase cannot happen.

1.3 Corollary *Let X be a minimal Kähler threefold. Assume that X is not both simple and non-Kummer. Then $\kappa(X) = \nu(X)$ and therefore K_X is semi-ample.*

As usual, $\nu(X)$ is the numerical Kodaira dimension of X , i.e. the largest number k such that $K_X^k \not\equiv 0$.

Proof. Again we may assume X not projective. By [Na87] it suffices to prove $\kappa(X) = \nu(X)$.

(1) We have $\kappa(X) \geq 0$ by [CP98].

(2) If $\kappa(X) = 0$, then $\nu(X) = 0$ by (1.1).

(3) If $\kappa(X) \geq 1$, we can apply [Ka85,7.3] to obtain $\kappa(X) = \nu(X)$; the arguments there remain valid in the Kähler case (at least for threefolds). Note that $K_X^3 > 0$ forces X to be projective.

1.4 Remark (1) The case that X is simple and not Kummer remains open. The first difficulty is that we might have K_X nef but $\kappa(X) = -\infty$. Furthermore, if $\kappa(X) = 0$, then at least $\nu(X) \neq 1$. In case X is projective, this is proved in [Mi88]; his arguments work in the Kähler case as well. However it might still be possible that there is a simple non-Kummer minimal threefold with $\kappa(X) = 0$ and $\nu(X) = 2$.

Everything would be settled if one could produce a 1-form on a minimal simple threefold, possibly after finite cover, étale in codimension 1 or 2. In fact, suppose that $q(X) > 0$ and X smooth for simplicity and later reference. Then, X being simple, we must have $q(X) = 3$, the Albanese $\alpha : X \rightarrow A$ has to be onto the three-dimensional torus A . Since A has no divisors, α is unramified in codimension 1, so that actually, arguing as in (1.1), α is bimeromorphic and X is Kummer.

(2) We treat one of the open problems in (1) in a special case. Assume that X is smooth and that $mK_X = \mathcal{O}_X(D)$ with a **smooth** surface (and X simple, K_X nef). By adjunction K_D is nef and $K_D^2 = 0$. Therefore $c_2(D) \geq 0$. On the other hand, $c_2(X) \cdot D = c_2(D)$ and $c_2(X) \cdot D = c_2(X) \cdot mK_X = -24m\chi(X, \mathcal{O}_X)$. If $q(X) = 0$, then $\chi(X, \mathcal{O}_X) > 0$, hence $c_2(D) < 0$, contradiction.

1.5 Theorem *Let X be a smooth compact Kähler threefold with $\kappa(X) = 0$. Suppose that X is not both simple and non-Kummer. Then X has a minimal model.*

Proof. The algebraic case being settled by Mori [Mo88] (and previously by Ueno and Viehweg if $q(X) > 0$, see [Vi80]), we may assume that X is not projective. Then however such a model has already been constructed in the proof of (1.1) (called \hat{X} there).

2. Existence of contractions

(2.1) Let X be a smooth compact Kähler threefold and $C \subset X$ a rational curve such that $K_X \cdot C < 0$. Then C determines a non-splitting family of rational curves [CP96]. In [CP96] and [Pe96] we proved that in turn a non-splitting family $(C_t)_{t \in T}$ of rational curves defines a contraction $\varphi : X \rightarrow Y$ except in one case. This case is the following, called Case (E) below:

X is non-projective, of course, $K_X \cdot C_t = -1$, the family (C_t) fills up a non-normal surface S , furthermore $\kappa(X) = -\infty$ and $S \cdot C_t \geq 0$. In this case we expect that X carries a

”generic” conic bundle structure, in particular X is uniruled and it is proved in [Pe96,3.5] that, given a generic conic bundle structure on X , then X carries a contraction.

The aim of this section is to settle Case (E), so that we obtain:

2.2 Theorem *Let X be a smooth compact Kähler threefold, $C \subset X$ a rational curve with $K_X \cdot C < 0$. Then there exists a surjective morphism $\varphi : X \rightarrow Y$ to a normal compact complex space Y such that*

- (1) φ has connected fibers;
- (2) $-K_X$ is φ -ample;
- (3) $\rho(X) = \rho(Y) + 1$;
- (4) Y is Kähler except possibly if Y is smooth and φ is blow-up along a smooth curve. In this last case Y is Kähler if and only if the class of a fiber is extremal in the closure of the dual of the Kähler cone.

Proof. As explained in (2.1), we have only to deal with the case (E). So assume that we are in the situation of (E).

(1) First we claim that it is sufficient to prove that X is uniruled. In fact, if X is uniruled, then [CP96,2.10] yields $a(X) \neq 2$ and the proof of ibid.(2.12) shows that $a(X) \neq 0$. Thus $a(X) = 1$. Now [CP96,2.13] shows that X carries a generic conic bundle structure (we used in (2.13) the assumption of algebraic approximability only to conclude uniruledness!). Hence a contraction exists as explained in (2.1).

(2) In case X is not simple, we can now just apply [CP98] to conclude from $\kappa(X) = -\infty$ that X is uniruled. However our reasoning below for the ”simple” case also applies here and makes the proof of (2.2) independent of [CP98]. In order to prove uniruledness we consider the normalisation $\nu : \tilde{S} \rightarrow S$. By [CP96] \tilde{S} is a smooth minimal ruled surface, so we have a \mathbf{P}_1 -bundle structure $g : \tilde{S} \rightarrow \tilde{C}$ over a smooth curve \tilde{C} . Let $N \subset S$ be the non-normal locus, equipped the conductor ideal and $\tilde{N} \subset \tilde{S}$ be the analytic preimage.

(4) We shall assume here that there is a multisection $C_1 \subset \tilde{N}$ of g and a fiber F of g such that $g(C_1) = g(F)$. We shall use the standard theory of ruled surfaces as treated in [Ha77,V.2]. Let C_0 be a section of \tilde{S} with C_0^2 minimal and let $e = -C_0^2$. Then we can write

$$C_1 = aC_0 + bF$$

with $a \geq 1$ (here $=$ means numerical equivalence). Consider the ”cycle map” $\nu_* : N_1(\tilde{S}) \rightarrow N_1(X)$. I claim that for every class $C' \in \overline{NE}(\tilde{S})$ there is a positive rational number μ such that

$$\nu_*(C') = \mu C_t = \mu \nu_*(F).$$

By our assumption

$$\nu_*(C_1) = \lambda \nu_*(F)$$

with a positive rational number λ . Hence

$$\nu_*(aC_0) = \nu_*(C_1) - b\nu_*(F) = (\lambda - b)\nu_*(F).$$

Let ω be a Kähler form on X . Then

$$\int_{C_0} \nu^*(\omega) = \left(\frac{\lambda - b}{a}\right) \int_{C_t} \omega > 0,$$

hence $\lambda > b$ and our claim (*) holds for $C' = C_0$. If now $e \geq 0$, then $\overline{NE}(\tilde{S}) = \mathbf{R}_+(C_0) + \mathbf{R}_+(F)$, and (*) follows. If however $e < 0$, then

$$\overline{NE}(\tilde{S}) = \mathbf{R}_+C' + \mathbf{R}_+F$$

with $C' = C_0 + \frac{e}{2}F$. Thus

$$\nu_*(C') = \nu_*(C_0) + \frac{e}{2}\nu_*(F) = \left(\frac{\lambda - b}{a} + \frac{e}{2}\right)\nu_*(F).$$

Integrating the Kähler form ω again, we see that the coefficient is positive, so we are done with (*). Our conclusion from (*) is the following:

(**) if L is a line bundle on S such that $L \cdot C_t > 0$ (resp. $L \cdot C_t = 0$) then $\nu^*(L)$ is ample, so L is ample (resp. $L \equiv 0$).

Since $S \cdot C_t \geq 0$, we deduce that either

- (a) the normal bundle N_S is ample, or
- (b) $N_S \equiv 0$.

In (a) we have $S^3 > 0$, hence X is projective [CP96,2.9].

(b) By [Mo82],[Re94], we have $H^1(S, \mathcal{O}_S) = 0$ since $-K_S$ is ample. In particular

$$\text{Pic}(S) \subset H^2(S, \mathbf{Z}).$$

We claim that

$$N_S \simeq \mathcal{O}_S. \tag{+}$$

In order to prove (+) we will verify that \tilde{N} consists only of C_0 and one fiber F so that in particular

$$H^1(\tilde{N}, \mathbf{Z}) = 0. \tag{++}$$

Assuming (++) for the moment, we consider the Mayer-Vietoris type sequence

$$H^1(\tilde{N}, \mathbf{Z}) \longrightarrow H^2(S, \mathbf{Z}) \longrightarrow H^2(\tilde{S}, \mathbf{Z}) \oplus H^2(N, \mathbf{Z}) \longrightarrow H^2(\tilde{N}, \mathbf{Z}).$$

Then (++) implies

$$H^2(S, \mathbf{Z}) \subset H^2(\tilde{S}, \mathbf{Z}) \oplus H^2(N, \mathbf{Z}) \simeq \mathbf{Z}^3$$

(notice that the reduction of N is an irreducible curve because \tilde{N} has only two components). Thus $H^2(S, \mathbf{Z})$ is torsion free. Moreover $c_1(N_S) = 0$ and therefore (+) follows from $\text{Pic}(S) \subset H^2(S, \mathbf{Z})$. From $H^1(\mathcal{O}_S) = 0$ and $H^2(S, \mathcal{O}_S) = H^0(S, \omega_S) = 0$ (note $\omega_S \cdot C_t = -1$), it follows

$$\chi(N_S) = \chi(\mathcal{O}_S) = 1,$$

hence S deforms in a 1-dimensional family whose general member S_t has again negative Kodaira dimension, i.e. $H^0(\omega_{S_t}^m) = 0$ for all positive m and therefore (going to a desingularisation) we see that S_t is uniruled, hence X is uniruled. It therefore remains to prove that \tilde{N} has only two components in order to finish (4). Write

$$\nu^*(\omega_S^*) = \alpha C_0 + \beta F$$

and

$$\tilde{N} = \gamma C_0 + \delta F.$$

Then $\alpha = 1$ (because $\nu^*(\omega_S^*) \cdot F = 1$) and moreover $\gamma > 0$ (because \tilde{N} contains C_1 , hence $\gamma \geq a$). We are going to use the equation

$$\omega_{\tilde{S}} = \nu^*(\omega_S) - \tilde{N}$$

(see [Mo82]). Since

$$\omega_{\tilde{S}}^* = 2C_0 + (e + 2 - 2g),$$

g the genus of C , we obtain

$$2 = \gamma + \alpha \text{ and } e + 2 - 2g = \beta + \delta.$$

Thus $\gamma = 1$. Notice that $\delta > 0$ since a fiber and a multi-section (now a section) are identified. Note also that $\nu^*(\omega_S^*)$ is ample by (**). Now it is a simple calculation using [Ha77, V.2] to obtain $g = 0, e \geq 0, \beta = e + 1$ and $\delta = 1$. So $\tilde{N} = C_0 + F$, even ideal-theoretically. This proves (+).

(5) Finally we have to treat the case that no multisection of g is identified with a fiber by ν . Then g induces a map $h : S \rightarrow B$ to a - usually non-normal - curve B and a map $\mu : C \rightarrow B$, the normalisation of C , such that

$$h \circ \nu = \mu \circ g.$$

So for every $t \in T$ we find some $b \in B$ such that $C_t \subset S_b := h^{-1}(b)$, and S_b consists only of C_t 's. We fix a general smooth point $b \in B$. Then S_b , the analytic fiber over b , is of the form

$$S_b = C_{t_1} + \dots + C_{t_r},$$

and of course S_b is as a Cartier divisor in a Gorenstein surface free from embedded points. Actually S_b is locally a complete intersection (in X).

(a) If S_b is reducible, then

$$-K_X \cdot S_b \geq 2,$$

hence by Ein's deformation lemma (see [Ko96, II.1.16]) the cycle S_b of rational curves deforms in an at least 2-dimensional family. Since the deformations of S_b inside S have dimension 1, we conclude the existence of a covering family (B_s) of curves for X (the general B_s might however not be rational). So X is not simple, the only case to be

excluded if we use [CP98]. Here is the way to avoid [CP98]: let $u : \tilde{S}_b = \nu^{-1}(S_b) \rightarrow S_b$ be the normalisation of S_b . Then \tilde{S}_b consists of r disjoint smooth rational curves. Now

$$\dim_{[u]} \mathrm{Hom}(\tilde{S}_b, X) \geq -K_X \cdot S_b + 3\chi(\mathcal{O}_{\tilde{S}_b}) = -K_X \cdot \tilde{S}_b + 3r,$$

see e.g. [Ko96]. Taking into account $\dim \mathrm{Aut}(\tilde{S}_b) = 3r$, we can still conclude the existence of a covering family of reducible rational curves for X so that X is uniruled.

(b) Suppose finally that S_b is irreducible. Having in mind that S_b is Cartier in S and also locally a complete intersection, we consider the exact sequence of conormal bundles

$$0 \rightarrow N_{S|X}^*|S_b \rightarrow N_{S_b|X}^* \rightarrow N_{S_b|S}^* \rightarrow 0.$$

Since $N_{S_b|S}^* = \mathcal{O}_{S_b}$ and since $N_{S|X}^*|S_b$ has nef dual, it follows via the sequence that $N_{S_b|X}$ is nef. This contradicts easily $-K_X \cdot C_t = -1$.

3. Existence of rational curves

In this section we prove that any smooth compact Kähler threefold X whose canonical bundle is not nef, carries a rational curve C with $K_X \cdot C < 0$, unless X is simple with negative Kodaira dimension. We first treat the case $\kappa(X) \geq 0$.

3.1 Theorem *Let X be a smooth compact Kähler threefold with $\kappa(X) \geq 0$. If K_X is not nef, there exists a rational curve $C \subset X$ with $K_X \cdot C < 0$, and therefore there is a contraction $\varphi : X \rightarrow Y$.*

Proof. Since $\kappa(X) \geq 0$, we find $m \in \mathbf{N}$ such that

$$mK_X = \mathcal{O}_X(\sum \lambda_i D_i),$$

where $\lambda_i \in \mathbf{N}$ and $D_i \subset X$ are irreducible. By [Pe96,4.9] there exists some irreducible component D_{i_0} such that $K_X|D_{i_0}$ is not nef. By renumbering we may assume that the components with this property are exactly D_1, \dots, D_r . Fix some $1 \leq i \leq r$ and put $D = D_i$. Let $\nu : \tilde{D} \rightarrow D$ be the normalisation and $\pi : \hat{D} \rightarrow \tilde{D}$ the minimal desingularisation. From adjunction we immediately obtain that

$$K_D = rK_X|D - E, \tag{*}$$

where $r > 1$ is a positive rational number and E an effective \mathbf{Q} -divisor. Next observe that

$$K_{\hat{D}} = \pi^* \nu^*(K_D) - E'$$

with another effective divisor, so that, putting $L = \pi^* \nu^*(K_X|D)$, we obtain a formula of \mathbf{Q} -divisors

$$L = aK_{\hat{D}} + A \tag{**}$$

with a positive rational number $a < 1$ and an effective \mathbf{Q} -divisor A .

(1) In a first step we construct an irreducible curve $C \subset D$ such that $K_X \cdot C < 0$. If $a(D) = 2$, the existence of C is clear, $K_X|_D$ being not nef.

(1.1) Suppose next $a(D) = 0$. Then \hat{D} is bimeromorphically a K3-surface or a torus. Then by (**), L has to be effective : there are curves $C_i \subset \hat{D}$ and positive integers p and m_i such that

$$pL = \sum m_i C_i.$$

Observe that every curve in \hat{D} is rational! So by [Pe96,4.9] we obtain a rational curve $C \subset \hat{D}$ with $L \cdot C < 0$, hence $K_X \cdot \nu(\pi(C)) < 0$. Thus in case $a(D) = 0$ for some component D we are already settled with the theorem.

(1.2) Now suppose $a(D) = 1$ and let $f : \hat{D} \rightarrow B$ be the algebraic reduction, an elliptic fibration. Since L is not nef, [Pe96,4.13] implies that either there exists an irreducible curve $C \subset \hat{D}$ with $L \cdot C < 0$ or $L = f^*(G)$ with G^* ample on B . This second alternative is clearly impossible since L is effective. Hence (1) holds also in this case.

(2) In all what follows we may assume C irrational, otherwise we are done. Now C deforms in an at least 1-dimensional family in X , therefore we can "extract" a maximal non-splitting family (C_t) of irreducible curves such that $K_X \cdot C_t < 0$. Here "maximal" means that no deformation of the general C_t splits. In other words, we can choose T to be (or rather project to) an irreducible component of the cycle space (which is automatically compact, X being Kähler). By $K_X \cdot C_t < 0$ all the C_t have to be in one of the D'_j s for $1 \leq j \leq r$, say in D . By (**), we have $K_D \cdot C_t < 0$. Let \hat{C}_t be the strict transform of a general C_t in \hat{D} . Then

$$L \cdot \hat{C}_t < 0.$$

In particular (**) yields $K_{\hat{D}} \cdot \hat{C}_t < 0$. \hat{C}_t being irrational, \hat{C}_t deforms in an at least 1-dimensional family, see [Ko96,II.1.14,1.15], and we can either extract a maximal non-splitting family (B_s) of irrational curves on \hat{D} such that

$$L \cdot B_s < 0$$

or we obtain the rational curve in D we are looking for. To be more precise, assume that we have a splitting. Take a splitting component C' with $L \cdot C' < 0$. If C' is irrational, then C' deforms in a family and we proceed as before (inductively). If C' is rational, then its image C'' in D is a rational curve with $K_X \cdot C'' < 0$ and we are done. Notice however that on surfaces X it is sometimes not possible to extract a non-splitting family from a splitting family (C_t) with $K_X \cdot C_t < 0$. The reason is that there are curves with $K_X \cdot C < 0$ which do not move, namely (-1) -curves, and these are the only ones.

Now we pass to a minimal model $\sigma : \hat{D} \rightarrow D_0$ and consider the induced family (B'_s) . Clearly (B'_s) is a non-splitting family. Suppose first that \hat{D} itself is not minimal, i.e. σ is not an isomorphism. Let $p \in D_0$ be a point blown up by σ . Then $p \in B'_s$ for some s . Since (B_s) is non-splitting, we must have $p \in B'_s$ for all s . Then Lemma 3.3 gives a contradiction.

So \hat{D} must be minimal. If \hat{D} is the projective plane, then clearly (B_s) has to be a family of lines. Anyway, then $\hat{D} = \tilde{D}$ and L^* is ample in that case, so the existence of C

rational with $K_X \cdot C < 0$ is clear. So \hat{D} is ruled with ruling $p : \hat{D} \rightarrow B$. By Lemma 3.4, its invariant $e < 0$, in particular \hat{D} has no exceptional curves and $\tilde{D} = \hat{D}$. Let F be a fiber of p and C_0 be a section with $C_0^2 = -e$ minimal. Let Z denote a class in the boundary component of $\overline{NE}(\hat{D})$ different from the ray generated by F . Then after rescaling

$$Z \equiv C_0 + \frac{e}{2}F.$$

Since $K_{\hat{D}} \cdot Z \geq 0$, we conclude from (***) easily that

$$L \cdot Z \geq 0$$

(consider curves near to the ray $\mathbf{R}_+(Z)$.) Using our assumption $L \cdot F \geq 0$ (otherwise we are done), we conclude that L is nef, contradicting $L \cdot \hat{C}_t < 0$.

Notice that in the second part of the proof of 3.1, where we deduced the existence of rational curve $C \subset X$ with $K_X \cdot C < 0$ from the existence of some curve C with $K_X \cdot C < 0$ we did not use the assumption $\kappa(X) \geq 0$. Therefore we have

3.2 Theorem *Let X be a smooth compact Kähler threefold. Assume that there is an irreducible curve $C \subset X$ with $K_X \cdot C < 0$. Then there is a rational curve $C \subset X$ with $K_X \cdot C < 0$ and hence there is a contraction on X .*

We still have to prove the following two lemmata.

3.3 Lemma *Let S be a ruled surface, $p \in S$. Let $(C_t)_{t \in T}$ be family of curves with C_t irreducible for general t and T compact (as always). Assume that $p \in C_t$ for all t . Then (C_t) splits.*

Proof. Assume that (C_t) does not split. Of course we may restrict ourselves to the case $\dim T = 1$. Let $f : S \rightarrow B$ be a ruling, F_b the fiber over $b \in B$. Since C_t cannot be a fiber, it must be a multi-section of f of degree d and therefore $C_t \cap F_b$ consists of d points, counted with multiplicity. Denoting by $\text{Hilb}_d(X)$ the Hilbert scheme of d points in a projective manifold X , we obtain for every b a map

$$\varphi_b : T \rightarrow \text{Hilb}_d(F_b).$$

Let $b_0 = f(p)$. Choose an open neighborhood U of b_0 in B and a trivialisation $S_U \simeq U \times \mathbf{P}_1$. Then we obtain a map

$$\Phi : U \times T \rightarrow \text{Hilb}_d(\mathbf{P}_1)$$

such that (by abuse of notation) $\Phi|_{\{b\} \times T} = \varphi_b$. Now notice that $\text{Hilb}_d(\mathbf{P}_1) = S^d \mathbf{P}_1$, the d -th symmetric product of \mathbf{P}_1 . Since all C_t pass through p , we conclude

$$\varphi_{b_0}(T) \subset \{p\} \times S^{d-1} \mathbf{P}_1 \subset S^d \mathbf{P}_1.$$

Using the identification $S^d \mathbf{P}_1 = \mathbf{P}_d$, the curve $\varphi_{b_0}(T)$ is contained in a hyperplane, i.e. degenerates. After shrinking U , there is no point $q \in F_b, b \in U, b \neq b_0$ such that all C_t pass through q . Therefore $\varphi_b(T)$ is not degenerate for $b \in U, b \neq b_0$. This leads to a contradiction since the $\varphi_b(T)$ form a family of curves: first if $d = 2$, then $\varphi_{b_0}(T)$ is a line, whereas $\varphi_b(T)$ is not for $b \neq b_0$; if $d \geq 3$, choose a generic projection $g : \mathbf{P}_d \rightarrow \mathbf{P}_2$ and consider the induced holomorphic maps $g \circ \varphi_b$.

3.4 Lemma *Let S be a ruled surface, $(C_t)_{t \in T}$ be a maximal (i.e. T projects to an irreducible component of the cycle space) non-splitting family of curves not consisting of fibers such that $K_S \cdot C_t < 0$. Then S comes from a stable vector bundle, i.e. its invariant e is negative.*

Proof. We write for numerical equivalence

$$C_t \equiv kC_0 + bF$$

where C_0 is a section with minimal C_0^2 and F is a fiber of f , a ruling. Again let $e = -C_0^2$. More precisely we write

$$\mathcal{O}_S(C_t) = \mathcal{O}_S(C_0) \otimes f^*(G_t)$$

with a line bundle G_t of degree b on B . By virtue of

$$K_S = -2C_0 + (2g - 2 - e)F$$

we obtain

$$K_S \cdot C_t = 2ke + k(2g - 2 - e) - 2b.$$

Hence $K_S \cdot C_t < 0$ translates into

$$b > \frac{ke}{2} + k(g - 1). \quad (*)$$

If now $e \geq 0$, i.e. E is not stable, then $(*)$ implies $b > g - 1$, hence Riemann-Roch gives $h^0(G_t) > 0$ and therefore we obtain a splitting inside the family.

We now turn to the case $\kappa(X) = -\infty$.

3.5 Theorem *Let X be a smooth compact Kähler threefold or a normal compact Kähler threefold which is \mathbf{Q} -factorial with at most terminal singularities. Assume $\kappa(X) = -\infty$ and X not simple. Then there exists a rational curve $C \subset X$ with $K_X \cdot C < 0$.*

Proof. Of course we have only to treat the non-projective case. By [CP98], X is uniruled. Hence our claim is clear, taking C to be a general member of a covering family of rational curves of X .

3.6 Remarks (1) In order to complete the picture that a compact Kähler threefold X with K_X not nef carries a rational curve C with $K_X \cdot C < 0$ it remains to prove that a

simple Kähler threefold with $\kappa(X) = -\infty$ carries some curve (not necessarily rational) with $K_X \cdot C < 0$. Of course this is a strange situation, which is expected not to exist. In (2) and (3) we treat some special cases.

(2) There is no simple compact Kähler manifold X with $\kappa(X) = -\infty$ such that $-K_X$ is hermitian semi-positive. In fact if X is simple, then $q(X) = 0$ and the same holds for every finite étale cover of X , hence [DPS96] implies $H^2(X, \mathcal{O}_X) = 0$, contradiction.

(3) The same should be true if $-K_X$ is only nef. We may clearly assume $q(X) = 0$. Since

$$\chi(X, -mK_X) = (2m + 1)\chi(X, \mathcal{O}_X)$$

we obtain a contradiction if we can prove that

$$h^2(-mK_X) \leq C \tag{*}$$

for a positive constant C and all positive integers m , at least if m is sufficiently divisible.

We finally summarize the results of this section:

3.7 Theorem *Let X be a smooth compact Kähler threefold with K_X not nef. Then there exists a rational curve $C \subset X$ with $K_X \cdot C < 0$ and therefore X carries a contraction unless (possibly) X is simple with $\kappa(X) = -\infty$.*

4. Kähler contact threefolds

Let X be a smooth compact Kähler threefold. We recall the notion of a contact structure on X and refer to [Le95] for details. Assume that there is a line bundle L with $-K_X = 2L$. A contact structure on X is an L -valued 1-form

$$\omega \in H^0(\Omega_X^1 \otimes L)$$

without zeroes such that locally $\omega \wedge d\omega$ is nowhere 0. We therefore have a vector bundle F of rank 2 and an exact sequence

$$0 \longrightarrow F \longrightarrow T_X \longrightarrow L \longrightarrow 0.$$

Ye [Ye94] proved that a projective contact threefold is either \mathbf{P}_3 or the projectivised tangent bundle of any projective surface. We extend this results to the Kähler case; however we cannot handle at the moment the possibility of a simple contact threefold with negative Kodaira dimension.

4.1 Theorem *Let X be a smooth compact Kähler threefold admitting a contact structure. Then X is one of the following*

- (1) $X = \mathbf{P}_3$;
- (2) X is the projectivised tangent bundle of a compact Kähler surface;

(3) X is simple with $\kappa(X) = -\infty$.

Of course case (3) should not exist. Note also that the projectivised tangent bundle of a Kähler surface has always a contact structure, see e.g. [Le94].

Proof. The projective case being proved in [Ye95], we shall assume X not projective. We may also assume that X is not simple with $\kappa(X) = -\infty$.

(1) First notice that the arguments in [Ye95] show that $\kappa(X) \leq 0$. In fact, the necessary results of Bogomolov are true in the Kähler case, too.

(2) Suppose that K_X is not nef. Then by (3.7) we find a contraction $\varphi : X \rightarrow Y$. Since K_X is divisible by 2, φ must be the blow-up of a smooth point or (analytically) a \mathbf{P}_1 -bundle over a Kähler surface. Assume first that φ is the blow-up of a point with exceptional divisor E . Using the notations from the beginning of the section, we have an exact sequence

$$0 \rightarrow N_E^* \otimes L \rightarrow \Omega_X^1 \otimes L|_E \rightarrow \Omega_E^1 \otimes L|_E \rightarrow 0.$$

Now $L|_E = \mathcal{O}_E(1)$, hence

$$H^0(\Omega_X^1 \otimes L|_E) = H^0(N_E^* \otimes L).$$

But since $N_E^* \otimes L = \mathcal{O}(2)$, every section in $\Omega_X^1 \otimes L$ must have zeroes, contradiction. Hence φ is a \mathbf{P}_1 -bundle over a Kähler surface Y and it follows as in [Ye95] that X is actually $\mathbf{P}(T_Y)$.

(3) It remains to treat the case that K_X is nef and $\kappa(X) = 0$. If X is not simple, then $K_X \equiv 0$ by (1.1). Now a finite étale cover of X is either a torus or a product of a K3-surface with an elliptic curve which immediately gives a contradiction. So we will suppose that X is simple. The contact form is a section in $H^0(X, \Omega_X^1 \otimes L)$ with $-K_X = 2L$. Hence we have an injection $L^* \rightarrow \Omega_X^1$. If we knew that $H^0(L^*) \neq 0$ then $q(X) \geq 1$ and X is a torus (see (1.1), (1.4)). However in general we know only that $\kappa(L^*) = \kappa(X) = 0$, i.e. $H^0(L^{*m}) \neq 0$ for some positive m . Then we take a generically finite map $f : \tilde{X} \rightarrow X$ such that

$$H^0(\tilde{X}, f^*(L^*)) \neq 0$$

and \tilde{X} is again smooth. Since $f^*(L^*) \subset f^*(\Omega_X^1)$ we obtain via the canonical injective map $f^*(\Omega_X^1) \rightarrow \Omega_{\tilde{X}}^1$ an injection

$$f^*(L^*) \subset \Omega_{\tilde{X}}^1$$

and therefore $q(\tilde{X}) > 0$. Now \tilde{X} must again be simple, in particular $\kappa(\tilde{X}) = 0$. Hence \tilde{X} is Kummer, more precisely the Albanese map $\alpha : \tilde{X} \rightarrow A$ is bimeromorphic onto the three-dimensional torus A . Therefore $K_{\tilde{X}} = B$ with B an effective divisor supported on the exceptional locus of α . On the other hand

$$K_{\tilde{X}} = f^*(K_X) + R$$

with another effective divisor R . Since K_X is nef, it follows easily that $R = B$ and thus $f^*(K_X) = \mathcal{O}_X$, i.e. $K_X \equiv 0$, or directly a covering $A \rightarrow X$. So X is Kummer.

4.2 Remark Assume that X is a simple threefold, $\kappa(X) = -\infty$ and X admits a contact structure. If there is a curve C with $K_X \cdot C < 0$, then by (3.2) we have a contraction $\varphi : X \rightarrow Y$. This is ruled out as in the proof of (4.1). Hence we can at least say that $K_X \cdot C \geq 0$ for all curves $C \subset X$.

4.3 Remark Observe that we have proved in (4.1) the following slightly more general statement.

Let X be a smooth compact Kähler threefold, $L \in \text{Pic}(X)$ such that $-K_X = 2L$. Let $\omega \in H^0(\Omega_X^1 \otimes L)$ be a section without zeroes. Then X is one the following

- (a) $X = \mathbf{P}_3$;
- (b) $X = \mathbf{P}(T_S)$, where S is a Kähler surface;
- (c) X is an étale "undercover" of a torus or a product of an elliptic curve with a K3-surface;
- (d) X is simple

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